International Journal of Theoretical Physics, Vol. 19, No. 6, 1980

Lorentz Invariance of Gravitational Lagrangians in the Space of Reference Frames

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Received November 21, 1979

The recently proposed theories of gravitation in the space of reference frames $\mathbb{S}$ are based on a Lagrangian invariant with respect to the homogeneous Lorentz group. However, in theories of this kind, the Lorentz invariance is not a necessary consequence of some physical principles, as in the theories formulated in space-time, but rather a purely esthetic request. In the present paper, we give a systematic method for the construction of gravitational theories in the space $\mathbb{S}$, without assuming a priori the Lorentz invariance of the Lagrangian. The Einstein–Cartan equations of gravitation are obtained requiring only that the Lagrangian is invariant under proper rotations and has particular transformation properties under space reflections and space-time dilatations.

1. INTRODUCTION

Theories of gravitation defined on a principal fiber bundle (Choquet-Bruhat, 1968; Kobayashi and Nomizu, 1969), with base the space-time manifold and fiber the Lorentz group, were introduced many years ago and largely discussed by several authors (Trautman, 1970, 1973; Cho, 1976; Mansouri and Chang, 1976). Their introduction becomes very natural if one considers gravitation as a special kind of gauge field theory (Utiyama, 1956; Kibble, 1961; Hehl et al., 1976). Recently, a Lagrangian formalism has been developed by Toller (1978) and independently by Ne'eman and Regge (1978a, b), dealing with similar and also more general theories.

In this formalism, the classical field theories are formulated in a space $\mathbb{S}$, whose points represent local inertial reference frames. The fields are functions of the points of $\mathbb{S}$, namely, they are functions of the reference frames. The motivation for taking this point of view was explained by Lurçat (1964) and by Toller (1975, 1977). In Section 2 we summarize some results given by Toller (1978, 1979) which we need.

In the space of reference frames, one can treat matter fields (Toller and Vanzo, 1978) as well as geometric fields describing gravitation or its generalizations (Toller et al., 1979) and Yang–Mills fields (Zerbini et al., 1978). The theories on $\mathbb{S}$ satisfying automatically the principle of relativity, because, in this space, it is not possible to distinguish a priori one point from the other; all the local inertial reference frames are equivalent. As a consequence of this fact, the conservation laws of energy, momentum, and angular momentum follow without assuming the invariance of the Lagrangian with respect to the Lorentz group.

All the Lagrangians on $\mathbb{S}$ studied until now and quoted above are Lorentz invariant. It seems to us that, at the moment, this symmetry is not justified from the operational point of view. In fact, the physical operations that define space translations are completely different from the ones that define time translations. In the same way, rotations and Lorentz boosts have a very different operational nature.

The aim of this paper is to build up a theory of gravitation without requiring the Lagrangian on $\mathbb{S}$ to be Lorentz invariant. To limit the possible choices, we shall assume that the Lagrangian is invariant with respect to the proper rotation group and has the behavior suggested by physical requirements under space reflections and space-time dilatations. The first assumption is justified by the fact that transformations of the reference frames, which are related by means of a rotation (for instance two space translations in different directions), have similar operational definitions. In Section 3 we introduce the symmetries of the Lagrangian and in Section 4 we treat in detail the special case of rotation-invariant Lagrangians. In Section 5 we see finally that the Lagrangian that is physically meaningful is Lorentz invariant and gives rise to the usual theory of gravitation in the presence of torsion. We must emphasize that, in order to obtain this result, we assume that for the vacuum solutions of the theory the space $\mathbb{S}$ can be identified with te Poincaré group. This does not mean that the Poincaré group is a symmetry group of the Lagrangian. In fact, there is no Poincaré-invariant Lagrangian with the required vacuum solutions. From this point of view, the result obtained is surprising.

2. SHORT RESUME OF THE GENERAL FORMALISM

In order to give a geometrical description of the gravitational phenomena, we consider the ten-dimensional space $\mathbb{S}$ of the local inertial reference frames. The geometry of $\mathbb{S}$ is described by ten vector fields $A_\alpha(\alpha = 0,\dots,9)$, linearly independent at every point of $\mathbb{S}$, which represent infinitesimal transformations of the reference frames, or by ten differential one-forms

ω^{α} defined by the relation

$$i_{\alpha}\omega^{\beta}=\delta_{\alpha}^{\beta} \tag{2.1}$$

where by i_{α} we indicate the inner product operator corresponding to the vector field A_{α}.

In this space the dynamics is described by an action principle of the kind

$$\delta\int_{S}\lambda=\delta\int_{S}\frac{1}{24}\lambda_{\alpha\beta\gamma\delta}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}=0 \tag{2.2}$$

where S is an arbitrary four-dimensional compact submanifold of $\mathbb{S}$ and λ is a differential 4-form, called Lagrangian form, whose coefficients $\lambda_{\alpha\beta\gamma\delta}$ are functions of the structure coefficients $F_{\alpha\beta}^{\gamma}$ defined by means of the following equation:

$$d\omega^{\gamma}=-\tfrac{1}{2}F_{\alpha\beta}^{\gamma}\omega^{\alpha}\wedge\omega^{\beta} \tag{2.3}$$

They are antisymmetric in the lower indices and satisfy the generalized Jacobi identity

$$L_{\alpha}F_{\beta\gamma}^{\delta}+L_{\beta}F_{\gamma\alpha}^{\delta}+L_{\gamma}F_{\alpha\beta}^{\delta}=F_{\alpha\beta}^{\eta}F_{\eta\gamma}^{\delta}+F_{\beta\gamma}^{\eta}F_{\eta\alpha}^{\delta}+F_{\gamma\alpha}^{\eta}F_{\eta\beta}^{\delta} \tag{2.4}$$

where by L_{α} we indicate the derivative along the direction individuated by the field A_{α}. Here and in the following, the sum over repeated indices is understood.

In the present paper we focus our attention on those theories that have the constant solution

$$F_{\alpha\beta}^{\gamma}=\hat{F}_{\alpha\beta}^{\gamma} \tag{2.5}$$

in an empty region of the space where the density and the flow of energy, momentum, and spin angular momentum of matter vanish. The quantities $\hat{F}_{\alpha\beta}^{\gamma}$ satisfy the Jacobi identity which follows from equation (2.4) and therefore they must be the structure constants of a Lie algebra. We suppose that they are the structure constants of the Poincaré algebra.

It has been shown (Toller, 1979) that the Lagrangian forms that provide the solution (2.5) must be of the kind

$$\lambda=\tfrac{1}{24}\left(F_{\alpha\beta}^{\theta}\hat{G}_{\theta\gamma\delta}-\tfrac{1}{2}\hat{F}_{\alpha\beta}^{\theta}\hat{W}_{\theta\gamma\delta}\right)\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta}+\lambda' \tag{2.6}$$

where λ' is any 4-form with a zero of order 2 at the point $F_{\alpha\beta}^{\gamma}=\hat{F}_{\alpha\beta}^{\gamma}$, $\hat{W}_{\alpha\beta\gamma}$, symmetric in the first two indices, are 550 arbitrary constants which have to satisfy the conditions

$$\hat{W}_{\alpha\beta\gamma} + \hat{W}_{\beta\gamma\alpha} + \hat{W}_{\gamma\alpha\beta} = 0 \tag{2.7}$$

$$\left(2\hat{F}^{\theta}_{\alpha\beta}\hat{W}_{\rho\theta\gamma} - \hat{F}^{\theta}_{\alpha\beta}\hat{W}_{\rho\gamma\theta} + 2\hat{F}^{\theta}_{\rho\beta}\hat{W}_{\alpha\theta\gamma}\right)\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma} = 0 \tag{2.8}$$

and finally $\hat{G}_{\alpha\beta\gamma}$ are 450 constants given by

$$\hat{G}_{\alpha\beta\gamma} = \tfrac{1}{3}\left(\hat{W}_{\alpha\beta\gamma} - \hat{W}_{\alpha\gamma\beta}\right) + \hat{A}_{\alpha\beta\gamma} \tag{2.9}$$

where $\hat{A}_{\alpha\beta\gamma}$ is the completely antisymmetric part of $\hat{G}_{\alpha\beta\gamma}$ and can be neglected or arbitrarily choosen, because it does not affect the field equations. Equations (2.7) and (2.8) are not sufficient to determine all the quantities $\hat{W}_{\alpha\beta\gamma}$ and therefore, even if we ignore λ', there are plenty of Lagrangian forms which provide the solution (2.5) in an empty space. The natural way to limit the possible choices is to impose particular transformation properties of the Lagrangian with respect to a symmetry group. In the next sections we shall see that, with a convenient choice of the group, the quantities $\hat{W}_{\alpha\beta\gamma}$ are unambiguously determined.

3. SYMMETRIES OF THE LAGRANGIAN FORM

This Section is devoted to the clarification of what we mean by symmetries of the Lagrangian in the space of reference frames. For this purpose, we consider a transformation of the type

$$\omega^{\alpha} \to C^{\alpha}_{\beta}\omega^{\beta} \tag{3.1}$$

where C^{α}_{β} is a nonsingular constant matrix belonging to a group $\mathcal{G}$. We assume that under this transformation the Lagrangian transforms according to

$$\lambda \to w(C)\lambda \tag{3.2}$$

where $w(C)$ is a one-dimensional representation of the group $\mathcal{G}$.

Here we are particularly interested in four kinds of transformations, which all belong to the group of automorphism of the Lie algebra of the Poincaré group, i.e., the group of the matrices C which leave the structure constants $\hat{F}^{\gamma}_{\alpha\beta}$ unchanged. In order to continue, some conventions are necessary. We use the double index Aa, where A and a take independently the values 1,2,3, to cover the set $1,\ldots,9$. Moreover, we choose the vector fields A_{Aa} in such a way that A_{A1} represent space translations, A_{A2} represent space rotations, and A_{A3} represent Lorentz boosts. Of course, A_0 represents time translations.

The transformations that we take into consideration are the space reflections defined by

$$\omega^0 \to \omega^0, \qquad \omega^{Aa} \to (-1)^a \omega^{Aa} \tag{3.3}$$

the space-time dilatations given by

$$\omega^0 \to l\omega^0, \qquad \omega^{A2} \to \omega^{A2}, \qquad \omega^{A1} \to l\omega^{A1}, \qquad \omega^{A3} \to \omega^{A3}, \qquad l \neq 0 \tag{3.4}$$

and finally the proper Lorentz group and the proper rotation group, which act on the forms ω^α according to the adjoint representation of the Poincaré group. The infinitesimal transformations are given by

$$\omega^\alpha \to \omega^\alpha + \varepsilon^\gamma \hat{F}^\alpha_{\gamma\beta} \omega^\beta \tag{3.5}$$

The infinitesimal parameters $\varepsilon^4, \varepsilon^5, \varepsilon^6$, namely, ε^{A2} describe the rotations, while the parameters $\varepsilon^7, \varepsilon^8, \varepsilon^9$, namely, ε^{A3} correspond to Lorentz boosts.

Under the transformations (3.3) and (3.4) the Lagrangian transforms according to

$$\lambda \to p\lambda, \qquad p = \pm 1 \tag{3.6}$$

$$\lambda \to l^d \lambda \tag{3.7}$$

respectively, and under the transformation (3.5) it remains unchanged. If we consider Lagrangian forms of the kind (2.6) with $\lambda' = 0$, we see that equation (3.6) is true if the number of indices α, β, γ of the constant quantities $\hat{W}_{\alpha\beta\gamma}$ or $\hat{G}_{\alpha\beta\gamma}$, of the type $A1$ or $A3$, is odd or even according to whether $p = -1$ or $p = +1$. In the same way, one sees that equation (3.7) is verified if the number of indices α, β, γ of the constants $\hat{W}_{\alpha\beta\gamma}$ or $\hat{G}_{\alpha\beta\gamma}$, which take the values $0, \ldots, 3$ is equal to d.

The theories of gravitation in the space of reference frames studied until now were built up by requiring the Lagrangian form to be invariant with respect to the proper Lorentz group and to change its sign under the space reflections (3.3). The last requirement is a necessary condition for having the total energy invariant under this operation. If we impose these constraints on the Lagrangian (2.6) and we choose $\lambda' = 0$, we obtain the Ne'eman Regge Lagrangian form (5.6), which gives rise to the Einstein–Cartan equations. Under the dilatations (3.4) it transforms according to equation (3.7) with $d = 2$, which is just the right behavior required by Newton's theory of gravitation, as we shall explain in detail in Section 5.

We recall that every infinitesimal transformation of the reference frame has an operational meaning, namely, it can be performed by means of precise physical operations, whose only purpose is to build up a new reference frame s infinitely near to a preexistent one s_0. We stress that the symmetry transformations defined in this section have a different meaning. They do not act on the space of reference frames but on the space of infinitesimal transformations. For instance, a rotation considered as an element of the symmetry group transforms an infinitesimal space translation into another one, along a different direction. In a similar way, it transforms an infinitesimal rotation into another one and a Lorentz boost into another Lorentz boost. Therefore, we see that a rotation transforms an infinitesimal transformation into another one, which has a similar operational definition. This remark justifies, as we said in the Introduction, the assumption that the Lagrangian is rotationally symmetric. On the contrary, a Lorentz boost, considered as a symmetry operation, transforms a given symmetry operator into another one, which has a completely different operational nature. After this analysis, it becomes natural to ask why one wants the Lagrangian invariant with respect to the whole Lorentz group. In fact, we give up this symmetry and keep only the rotational invariance, which still seems to us physically justified.

4. ROTATION-INVARIANT LAGRANGIAN FORMS

Following the program outlined in the Introduction, now we study the most general rotationally invariant Lagrangian form of the kind (2.6) with $\lambda' = 0$. For this aim, we build up the constant quantities $\hat{W}_{\alpha\beta\gamma}$ in such a way that they are invariant with respect to the proper rotations and satisfy equations (2.7) and (2.8). The only constant ingredients that we can use are the Kronecker symbol δ_{AB} and the completely antisymmetric Levi–Civita symbol e_{ABC}. Then, the nonvanishing components of $\hat{W}_{\alpha\beta\gamma}$ can be written in the form

$$\hat{W}_{0AaBb} = \hat{W}_{Aa0Bb} = w_{0ab}\delta_{AB} \tag{4.1}$$

$$\hat{W}_{AaBb0} = \hat{W}_{BbAa0} = w_{ab0}\delta_{AB} \tag{4.2}$$

$$\hat{W}_{AaBbCc} = \hat{W}_{BbAaCc} = w_{abc}e_{ABC} \tag{4.3}$$

where $w_{0ab} = w_{a0b}$, $w_{ab0} = w_{ba0}$, and $w_{abc} = -w_{bac}$ are 24 arbitrary constants. From equation (2.7) we see that the number of independent ones decreases to 17; we obtain in fact the seven relations

$$w_{0ab} + w_{0ba} + w_{ab0} = 0 \tag{4.4}$$

$$w_{abc} + w_{bca} + w_{cab} = 0, \qquad a \neq b \neq c \tag{4.5}$$

If we want our theory to have the solution (2.5) in an empty space, the quantities $\hat{W}_{\alpha\beta\gamma}$ must satisfy equation (2.8). In order to solve this equation it is useful to write the nonvanishing structure constants $\hat{F}^{\gamma}_{\alpha\beta}$ of the Poincaré group in the following manner:

$$\hat{F}^{0}_{AaBb} = -\hat{F}^{0}_{BbAa} = f^{0}_{ab}\delta_{AB} \tag{4.6}$$

$$\hat{F}^{Aa}_{0Bb} = -\hat{F}^{Aa}_{Bb0} = f^{a}_{0b}\delta_{AB} \tag{4.7}$$

$$\hat{F}^{Aa}_{BbCc} = -\hat{F}^{Aa}_{CcBb} = f^{a}_{bc}e_{ABC} \tag{4.8}$$

with

$$f^{0}_{31} = -f^{0}_{13} = 1 \tag{4.9}$$

$$f^{1}_{30} = -f^{1}_{03} = 1 \tag{4.10}$$

$$f^{1}_{12} = f^{1}_{21} = f^{2}_{22} = f^{3}_{23} = f^{3}_{32} = -f^{2}_{33} = 1 \tag{4.11}$$

All the other coefficients vanish. A straightforward, but tedious calculation provides 11 other independent relations between the constants w_{0ab}, w_{ab0}, and w_{abc}. The final result is given by

$$\begin{aligned} &w_{131} = -w_{311} = A \\ &w_{120} = w_{210} = -A \\ &w_{021} = w_{201} = A \end{aligned} \tag{4.12}$$

$$\begin{aligned} &w_{121} = -w_{211} = B \\ &w_{130} = w_{310} = B \\ &w_{031} = w_{301} = -B - C \\ &w_{013} = w_{103} = C \end{aligned} \tag{4.13}$$

$$\begin{aligned} &w_{133} = -w_{313} = D \\ &w_{230} = w_{320} = -D \\ &w_{023} = w_{203} = D \end{aligned} \tag{4.14}$$

$$\begin{aligned} &w_{123} = -w_{213} = -E \\ &w_{330} = -2E \\ &w_{033} = w_{303} = E \end{aligned} \tag{4.15}$$

$$w_{233} = -w_{323} = F \tag{4.16}$$

where A B, C, D, E, and F are arbitrary constants. We have grouped together the quantities which induce different rules of transformation on the relative Lagrangian under equations (3.3) and (3.4). From these equations we see that there is a large number of nonvanishing rotational-invariant quantities $\hat{W}_{\alpha\beta\gamma}$, which provide a theory with the solution (2.5) in an empty space. Many of these choices give rise to theories that have unphysical solutions in an empty space and in the presence of matter. We must therefore impose other constraints on the Lagrangian form which limit the possible choices of $\hat{W}_{\alpha\beta\gamma}$.

5. GRAVITATIONAL LAGRANGIAN FORM

In Section 3 we said that, in order to obtain the total energy invariant under the space reflections, the Lagrangian has to change its sign under this transformation. We do not want to give up this property of energy and therefore we take $p=-1$ in equation (3.6). This means that the number of indices α, β, γ of the quantities $\hat{W}_{\alpha\beta\gamma}$ of the type $A1$ and $A3$ must be odd. In Table I we put the parameters A, B, C, D, E, and F which can be different from zero if p has a given value.

Since we always have Newton's theory of gravitation in our minds, we require that the Lagrangian form has the correct behavior suggested by this theory, under space-time dilatations. In order to obtain the value of d in equation (3.7), we resort to a simple dimensional reasoning. For simplicity, we use a units system in which the speed of light is equal to 1. We indicate the gravitational constant by κ; its dimensions, given by Newton's theory, are

$$[\kappa]=[\text{mass}]^{-1}[\text{length}] \tag{5.1}$$

The Lagrangian form λ has the dimensions of an action. We write it in the form

$$\lambda=\frac{1}{\kappa}\lambda_{\alpha\beta\gamma\delta}\omega^{\alpha}\wedge\omega^{\beta}\wedge\omega^{\gamma}\wedge\omega^{\delta} \tag{5.2}$$

TABLE I

$d \backslash p$	$+1$	-1
0	F	—
1	E	D
2	B, C	A
3	—	—

where now $\lambda_{\alpha\beta\gamma\delta}$ are its adimensional coefficients. Since the differential forms ω^{A2} and ω^{A3} are adimensional according to their meaning, and ω^0 and ω^{A1} have the dimensions of a length, we must have

$$[\lambda]=[\text{mass}][\text{length}]=\left[\frac{1}{\kappa}\right][\text{length}]^d \tag{5.3}$$

where d is the number of indices $\alpha,\beta,\gamma,\delta$ which take the values $0,\ldots,3$ and is the same as in equation (3.7). Then we get $d=2$. In Table I we put the parameters which can be different from zero when d has a given value. We see that only the constants $\hat{W}_{\alpha\beta\gamma}$ which derive from equation (4.12) have both the required properties $p=-1$ and $d=2$. They depend only on the parameter A.

Now, it is very easy to build up the Lagrangian (2.6) with the constant quantities $\hat{W}_{\alpha\beta\gamma}$ which depend only on the parameter A. Of course, the Lagrangian that one obtains is invariant with respect to the proper rotation group and transforms according to equation (3.6) with $p=-1$ and equation (3.7) with $d=2$ under space reflections and space-time dilatations, respectively. If we choose $A=6/\kappa$ and calculate the constants $\hat{W}_{\alpha\beta\gamma}$ according to equations (4.1)–(4.3) we have

$$\hat{W}_{ars}=\hat{W}_{ras}=\frac{3}{\kappa}\hat{F}^i_{al}g^{lj}e_{ijrs} \tag{5.4}$$

where now the Latin letters $a,b,\ldots,h$ can take the values $4,\ldots,9$ and the letters $i,j,k\ldots$ can take the values $0,\ldots,3$. By means of equation (2.9), we can calculate also $\hat{G}_{\alpha\beta\gamma}$. If we choose the completely antisymmetric quantities $\hat{A}_{ars}=\frac{1}{3}\hat{W}_{ars}$, we get

$$\hat{G}_{ars}=\frac{3}{\kappa}\hat{F}^i_{al}g^{lj}e_{ijrs} \tag{5.5}$$

From equations (5.4), (5.5), and (2.6) with $\lambda'=0$ we obtain the Lagrangian form

$$\lambda=\frac{1}{8\kappa}\left(F^a_{\alpha\beta}-\hat{F}^a_{\alpha\beta}\right)\hat{F}^i_{al}g^{lj}e_{ijrs}\omega^\alpha\wedge\omega^\beta\wedge\omega^r\wedge\omega^s \tag{5.6}$$

which is the one proposed by Ne'eman and Regge (1978) and gives the Einstein–Cartan theory of gravitation (Hehl et al., 1976). As one can see immediately, this Lagrangian is Lorentz invariant.

In conclusion, we have shown that, in order to describe gravitational phenomena, it is not necessary to assume a priori the Lagrangian form to be invariant with respect to the Lorentz group. We got in fact the usual theory of gravitation in the presence of torsion, assuming, besides the existence of the Poincaré vacuum solution (2.5), the rotational invariance and some particular transformation properties, which are justified by physical requirements, with respect to space reflections and space-time dilatations. The Lorentz invariance of the Lagrangian is an automatic consequence of these constraints.

ACKNOWLEDGMENT

It is a great pleasure to thank M. Toller for continuous helpful discussions and suggestions.

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